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دراسة عددية لحلّ نموذج من مسائل القيم الابتدائية المركبة الضبابية الضعيفة المتجانسة الخطية ذات الأمثال الثابتة من المرتبة الثانية عددياً

أ.د. سليمان محمود*

أ.د. محمد علي**

لما رزوق***

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□ ملخّص □

يهدف هذا العمل إلى دراسة حل مسائل القيم الابتدائية– المركبة الضبابية الضعيفة من المرتبة الثانية عددياً لأول مرة. باستخدام دالة تحويل خاصة، نجد أنّ كلّ مسألة قيمة ابتدائية مركبة ضبابية ضعيفة من المرتبة الثانية (WFC-IVP) يمكن كتابتها كمسألتين كلاسيكيتين من مسائل القيم الابتدائية الحقيقية من المرتبة الثانية. و كلّ مسألة قيمة ابتدائية من المرتبة الثانية يمكن أن تكتب على شكل مسألة قيمة ابتدائية من نظام من المعادلات التفاضلية من المرتبة الأولى، ونطبّق "طريقة أولر" لإيجاد حلولها التقريبية. ثمَّ من حلول كلا المسألتين الكلاسيكيتين، نشكل حلول مسألتنا (مسألة القيمة الابتدائية– المركبة الضبابية الضعيفة). حيث يتم التركيز على المعادلة التفاضلية الخطية المتجانسة ذات الأمثال الحقيقية الثابتة مع شروط ابتدائية حقيقية و خطوة ثابتة. وبناءً عليه، استخدمنا برنامج البايثون لإيجاد النتائج العددية من خلال مثال و عرض التقريبات و الأخطاء المطلقة في جداول و توضيح الرسوم البيانية التي تمثلها.

الكلمات المفتاحية: الأعداد المركبة الضبابية الضعيفة، الدوال المركبة الضبابية الضعيفة، مسائل القيم الابتدائية الضبابية الضعيفة.

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*أستاذ في قسم الرياضيات، كلية العلوم، جامعة اللاذقية، اللاذقية، سوريا.
E-mail: suliman_mmn@yahoo.com, ORCID: 0009-0007-5638-7346.
**أستاذ في قسم الرياضيات، كلية العلوم، جامعة اللاذقية، اللاذقية، سوريا.
E-mail: mohamadAli@gmail.com, ORCID: 0009-0000-0283-0719.
طالبة دراسات عليا (دكتوراه)، قسم الرياضيات، كلية العلوم، جامعة اللاذقية، اللاذقية، سوريا.***
E-mail: lamarazouk94@gmail.com, ORCID: 0009-0000-6335-8446.

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A Numerical Study for Solving a Model of Second-Order Real Constant-Coefficient Linear Homogeneous Weak Fuzzy Complex Initial Value Problems (WFC-IVPs)

Suliman Mahmoud*
Mohamad Ali**
Lama Razouk***

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□**ABSTRACT** □

This work aims to solve for the first time the second-order Weak Fuzzy Complex- Initial Value Problem (WFC-IVP) numerically. Using a special isomorphic transformation function, a second-order WFC-IVP can be written as two classical second-order initial Value Problems (IVPs) with respect to their own real variables. Where each second-order IVP can be expressed as an initial value problem for a system of first-order differential equations, and we use Euler's method to find approximate solutions. Then, we construct the numerical solutions for our problem (WFC-IVP) from the approximate solutions of related classical systems. We focus on a model of the second-order real constant-coefficient linear homogeneous differential equation with real initial conditions and fixed step. Therefore, we use Python to obtain numerical results for related example, where approximations and absolute errors are shown in tables and diagrams.

Keywords: Weak Fuzzy Complex numbers, Weak Fuzzy Complex functions, Weak Fuzzy Complex Initial Value Problem (WFC-IVP).

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*Professor, Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria.
E-mail: suliman_mmn@yahoo.com, ORCID: 0009-0007-5638-7346.
**Professor, Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria.
E-mail: mohamadAli@gmail.com, ORCID: 0009-0000-0283-0719.
Syria, 11.5.2025
***A PhD student, Department of Mathematics, Sciences College, Latakia University, Latakia, Syria.
E-mail: lamarazouk94@gmail.com, ORCID: 0009-0000-6335-8446

1 Introduction

In 2023, the Weak Fuzzy Complex Numbers (WFC- Numbers) was defined for the first time in [1] as a new generalization of classical real numbers. Researchers studied vector spaces on the new set of WFC numbers [2], and matrices [3]. Also, in [4–6] Weak Fuzzy Complex Diophantine equations have been discussed. Foundations of number theory were built in [7], and number theoretical concepts in the set of Weak Fuzzy Complex integers.

In [8, 9], the geometrical solutions for some vectorial equations are important applications of the WFC set known as A-Curves. However, WFC numbers and some of their arithmetic operations are shown using Python and Jupyter Notebook in [10].

Additionally, [11] introduced a special isomorphism transformation function and presented the Weak Fuzzy Complex functions. Then, [12] defined the Weak Fuzzy Complex Ordinary Differential Equations (WFC-ODEs).

1.1 Importance and Aim of This Research

The importance of this research appears with the need to discover the Initial Value Problems with their weak fuzzy complex variables. The main purpose of the paper is to solve a second-order constant-coefficient linear homogeneous Weak Fuzzy Complex- initial value problem (WFC-IVP) numerically.

1.2 Methodology

Theoretical part: We discuss numerical solving the second-order constant-coefficient linear homogeneous Weak Fuzzy Complex- initial value problems (WFC-IVPs), which becomes two systems of first-order IVPs in $\boldsymbol{R}$ using Euler's method.

Practical part: Numerical experiments are presented with tables of approximate solutions and absolute errors that illustrate the theoretical results. We have accomplished the computations using Python and Jupyter Notebook.

1.3 Paper Outline

The paper is organized as follows: we will mention some main definitions about Weak Fuzzy Complex numbers, a special isomorphism transformation function, WFC-functions, and WFC-ODEs in **section 2.** Then, in **section 3**, we will introduce the steps to solve the second-order real constant-coefficient linear homogeneous Weak Fuzzy Complex-Initial Value Problems numerically, where we will use Euler's method to get the approximations. In **section 4**, we will focus on the case of real initial conditions and fixed step for our WFC-IVP. Also, we will present an example to illustrate with tables and diagrams of approximate solutions and absolute errors that we will get using Python (the main instructions are included in the **Appendix**).

2 Preliminaries

Some important concepts will be mentioned in this section to understand our discussion:

Definition 1.[1] The *set of Weak Fuzzy Complex numbers* was defined as follows,

$\boldsymbol{F}_J = \{x_0 + x_1 J\,;\ x_0, x_1 \in \boldsymbol{R}, J^2 = t \in]0,1[\,\}$,

where 'J' is the Weak Fuzzy Complex operator ($J \notin \boldsymbol{R}$).

Definition 2. [11] Let φ be *the transformation function* from $\boldsymbol{F}_J$ to $\boldsymbol{R} \times \boldsymbol{R}$, which we define as follows:

$\varphi: \boldsymbol{F}_J \mapsto \boldsymbol{R} \times \boldsymbol{R}.$

$\varphi(x_0 + x_1 J) = (x_0 + x_1(-\sqrt{t}), x_0 + x_1(+\sqrt{t})) = (x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t})$;

where $J^2 = t \in]0,1[\Rightarrow J = \pm\sqrt{t}$, and $x_0, x_1 \in \boldsymbol{R}$ (This map is an isomorphism).

Definition 3. [11] Let $\varphi: \boldsymbol{F}_J \mapsto \boldsymbol{R} \times \boldsymbol{R}$ such that: $\varphi(X) = (a, b)$, the *inverse function* of φ is defined as follows:

$\varphi^{-1}: \boldsymbol{R} \times \boldsymbol{R} \mapsto \boldsymbol{F}_J$

$\varphi^{-1}(a,b) = \frac{1}{2}(a+b) + \frac{1}{2\sqrt{t}} J(b-a).$

Definition 4.[11]Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in \boldsymbol{F}_J$, we say that $X \leq Y$, if and only if:

$$\begin{cases} x_0 - x_1\sqrt{t} \leq y_0 - y_1\sqrt{t} \\ x_0 + x_1\sqrt{t} \leq y_0 + y_1\sqrt{t} \end{cases};\ x_0, x_1, y_0, y_1 \in \boldsymbol{R}.$$

Definition 5. [11] Let $A = a_0 + a_1 J$, $B = b_0 + b_1 J \in \boldsymbol{F}_J$, we define *the interval* $[A, B]$ if and only if $A \leq B$, according to the definition of the partial order relation ($\leq$).

- If $A \nleq B$, then $[A, B] = \phi$.
- We can understand $[A, B]$ as follows:

$[A, B] = \{C \in \boldsymbol{F}_J : C = c_0 + c_1 J \in \boldsymbol{F}_J;\ A \leq C \leq B\}.$

Definition 6. [11] Let $f: \boldsymbol{F}_J \mapsto \boldsymbol{F}_J$ be a Weak Fuzzy Complex *function* in one variable, where

$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}));\ f_1, f_2: \boldsymbol{R} \mapsto \boldsymbol{R}$,

then we say:

1) f is *continuous* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are continuous on $\boldsymbol{R}$.

2) f is *differentiable* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are differentiable on $\boldsymbol{R}$, with respect to their own variables.

3) f is *integrable* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are integrable on $\boldsymbol{R}$.

Definition 7. [11] Let $f: \boldsymbol{F}_J \mapsto \boldsymbol{F}_J$ be a *differentiable/integrable* function on $\boldsymbol{F}_J$. We define

1) $f'(X) = \varphi^{-1}\big(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})\big).$

2) $\int f(X).dX = \varphi^{-1}\big(\int f_1.d(x_0 - x_1\sqrt{t}), \int f_2.d(x_0 + x_1\sqrt{t})\big).$

Definition 8.[12] *The weak fuzzy complex differential equation of the second order* is written as follows:

$\mathcal{F}(X, Y, Y', Y'') = 0$ **(1)**

$$\overset{\varphi}{\Leftrightarrow}\begin{cases}\mathcal{F}_1(x_0-x_1\sqrt{t},y_0-y_1\sqrt{t},{y'}_0-{y'}_1\sqrt{t},{y''}_0-{y''}_1\sqrt{t})=0 & (2-1)\\ \mathcal{F}_2(x_0+x_1\sqrt{t},y_0+y_1\sqrt{t},{y'}_0+{y'}_1\sqrt{t},{y''}_0+{y''}_1\sqrt{t})=0 & (2-2)\end{cases}$$

(2)

where $\mathcal{F}=\varphi^{-1}(\mathcal{F}_1,\mathcal{F}_2)$, $y_0-y_1\sqrt{t}=f_1(x_0-x_1\sqrt{t})$, $y_0+y_1\sqrt{t}=f_2(x_0+x_1\sqrt{t})$

$Y'=\frac{dY}{dX}=f'(X)=\varphi^{-1}\big({f'}_1(x_0-x_1\sqrt{t}),{f'}_2(x_0+x_1\sqrt{t})\big)$,

$Y''=\frac{d^2Y}{dX^2}=f''(X)=\varphi^{-1}\big({f''}_1(x_0-x_1\sqrt{t}),{f''}_2(x_0+x_1\sqrt{t})\big)$,

and $\frac{d}{dX}=\varphi^{-1}\left(\frac{d}{d(x_0-x_1\sqrt{t})},\frac{d}{d(x_0+x_1\sqrt{t})}\right)$, $\frac{d^2}{dX^2}=\varphi^{-1}\left(\frac{d^2}{d(x_0-x_1\sqrt{t})^2},\frac{d^2}{d(x_0+x_1\sqrt{t})^2}\right)$.

Definition 9.[12] Let $y_0-y_1\sqrt{t}=f_1(x_0-x_1\sqrt{t})$ and $y_0+y_1\sqrt{t}=f_2(x_0+x_1\sqrt{t})$ are the general solutions to (2-1) on $\boldsymbol{I}_1\subseteq\boldsymbol{R}$ and (2-2) on $\boldsymbol{I}_2\subseteq\boldsymbol{R}$, respectively, then

$Y=f(X)=\varphi^{-1}\big(f_1(x_0-x_1\sqrt{t}),f_2(x_0+x_1\sqrt{t})\big)$ is *the general solution* of (1) on $\boldsymbol{I}=\varphi^{-1}(\boldsymbol{I}_1\times\boldsymbol{I}_2)\subseteq\boldsymbol{F}_J$,

where $X=\varphi^{-1}(x_0-x_1\sqrt{t},x_0+x_1\sqrt{t})$, $x_0-x_1\sqrt{t}\in\boldsymbol{I}_1$, $x_0+x_1\sqrt{t}\in\boldsymbol{I}_2$.

One of the simplest types of second-order ordinary differential equations is the linear homogeneous equation with constant coefficients. We will concentrate on this type which occurs in many applications in science and engineering.

Definition 10. The general form of a *second-order linear homogeneous weak fuzzy complex ordinary differential equation with real constant coefficients* is written as,

$aY''+bY'+cY=0,$ **(3)**

when the coefficients are real a, b and $c\in\boldsymbol{R}$, using φ, we find that (3) is equivalent to two second-order linear homogeneous ordinary differential equation with constant coefficients in $\boldsymbol{R}$,

$$aY''+bY'+cY=a\,\varphi^{-1}({Y_0}'',{Y_1}'')+b\varphi^{-1}({Y_0}',{Y_1}')+c\varphi^{-1}(Y_0,Y_1)$$
$$=\varphi^{-1}(aY_0''+bY_0'+cY_0,\quad a{Y_1}''+b{Y_1}'+cY_1)=0$$

$$\Leftrightarrow\begin{cases}a{Y_0}''+b{Y_0}'+cY_0=0\\ a{Y_1}''+b{Y_1}'+cY_1=0\end{cases}$$

where $Y_0=y_0-y_1\sqrt{t}\in\boldsymbol{R},Y_1=y_0+y_1\sqrt{t}\in\boldsymbol{R},Y=\varphi^{-1}(Y_0,Y_1)\in\boldsymbol{F}_J$.

Definition 11. The *real constant-coefficient second-order linear homogeneous weak fuzzy complex initial value problem* (WFC-IVP) is formed as,

$$\begin{cases}aY''+bY'+cY=0\\ Y(\alpha)=\nu,Y'(\alpha)=\omega\end{cases};\alpha\le \mathrm{X}\le\beta \qquad ,a,b,c\in\boldsymbol{R} \quad \textbf{(4)}$$

where

$Y=\varphi^{-1}(Y_0,Y_1),X=\varphi^{-1}(X_0,X_1)\in\boldsymbol{F}_J$,

$\alpha = \varphi^{-1}(\alpha_0, \alpha_1), \beta = \varphi^{-1}(\beta_0, \beta_1), \nu = \varphi^{-1}(\nu_0, \nu_1),$
$\omega = \varphi^{-1}(\omega_0, \omega_1) \in \boldsymbol{F}_J,$
$\alpha_0, \alpha_1, \beta_0, \beta_1, \nu_0, \nu_1, \omega_0, \omega_1 \in \boldsymbol{R}$

Now, we will focus on finding the numerical solutions of this kind of problems.

3 Numerical Solving of the Second-Order Real Constant-Coefficient Linear Homogeneous Weak Fuzzy Complex Initial Value Problems (WFC-IVPs)

To find the numerical solutions of the real constant-coefficient second-order linear homogeneous WFC-IVP (4), we follow the following steps:

<u>First step:</u> Write the equivalent IVPs in *R*.

We know that (4) is equivalent to (5) and (6) using φ,

$$\begin{cases} \begin{cases} aY_0'' + bY_0' + cY_0 = 0 \\ Y_0(\alpha_0) = \nu_0, Y_0'(\alpha_0) = \omega_0 \end{cases} ; \alpha_0 \le X_0 \le \beta_0 & (5) \\ \begin{cases} aY_1'' + bY_1' + cY_1 = 0 \\ Y_1(\alpha_1) = \nu_1, Y_1'(\alpha_1) = \omega_1 \end{cases} ; \alpha_1 \le X_1 \le \beta_1 & (6) \end{cases}$$

where their analytical solutions $\begin{cases} Y_0 = \tilde{H}e^{3X_0} + \tilde{I}e^{X_0} \\ Y_1 = \tilde{G}e^{3X_1} + \tilde{K}e^{X_1} \end{cases}$; $\tilde{H}, \tilde{I}, \tilde{G}, \tilde{K}$ are determined constants, and

$X = \varphi^{-1}(X_0, X_1) \in [\alpha, \beta] \subseteq \boldsymbol{F}_J,$ $X_0 \in [\alpha_0, \beta_0] \subseteq \boldsymbol{R}, X_1 \in [\alpha_1, \beta_1] \subseteq \boldsymbol{R},$

$Y = \varphi^{-1}(Y_0, Y_1), \alpha = \varphi^{-1}(\alpha_0, \alpha_1), \beta = \varphi^{-1}(\beta_0, \beta_1), \nu = \varphi^{-1}(\nu_0, \nu_1),\ \omega = \varphi^{-1}(\omega_0, \omega_1).$

<u>Second step:</u> Transform each second-order IVP into a system of first-order IVPs [13, 14].

Suppose $Z_0 = Y_0',\ Z_1 = Y_1',$

$$\begin{cases} \begin{cases} Y_0' = Z_0 \\ Z_0' = \dfrac{-b}{a}Z_0 - \dfrac{c}{a}Y_0 \\ Y_0(\alpha_0) = \nu_0, Z_0(\alpha_0) = \omega_0 \end{cases} ; \alpha_0 \le X_0 \le \beta_0 & (7) \\ \begin{cases} Y_1' = Z_1 \\ Z_1' = \dfrac{-b}{a}Z_1 - \dfrac{c}{a}Y_1 \\ Y_1(\alpha_1) = \nu_1, Z_1(\alpha_1) = \omega_1 \end{cases} ; \alpha_1 \le X_1 \le \beta_1 & (8) \end{cases}$$

<u>Third step:</u> Use a numerical method to approximate the solutions.

The systems (7) and (8) can be solved numerically by simply applying a particular numerical method. In each system, each member is treated separately but simultaneously,

$$\begin{cases} at\ X_0[i] \begin{cases} S_0[i]\ the\ approximations\ to\ Y_{0i} = Y_0[i] \\ T_0[i]\ the\ approximations\ to\ Y_{0i}' = Y_0'[i] \end{cases} \\ at\ X_1[i] \begin{cases} S_1[i]\ the\ approximations\ to\ Y_{1i} = Y_1[i] \\ T_1[i]\ the\ approximations\ \text{to}\ Y_{1i}' = Y_1'[i] \end{cases} \end{cases} ; i = 0,1, \dots, N.$$

Then, the outputs are $\begin{cases}(X_1[i], S_1[i], T_1[i]) \\ (X_0[i], S_0[i], T_0[i])\end{cases} \quad ; i = 0,1, \dots, N.$

<u>Fourth step:</u> Construct the final numerical solutions from the numerical solutions in the previous step using the transformation function.

We use the special transformation function φ^{-1},
$S[i] = \varphi^{-1}(S_0[i], S_1[i])$ the approximations to $Y_i = Y[i]$,
$T[i] = \varphi^{-1}(T_0[i], T_1[i])$ the approximations to $Y_i{}' = Y'[i]$,
at $X[i] = \varphi^{-1}(X_0[i], X_1[i])$.
where Y_i −the exact solution of the WFC-IVP.
Then, we have the following absolute errors:
$e[i] = \varphi^{-1}(e_0[i], e_1[i]) = \varphi^{-1}(|Y_{0i} - S_0[i]|, |Y_{1i} - S_1[i]|\ ),$
$E[i] = \varphi^{-1}(E_0[i], E_1[i]) = \varphi^{-1}(|Y_{0i}{}' - T_0[i]|, |Y_{1i}{}' - T_1[i]|).$

In our work, we will use Euler's method which is one of the most simple and famous numerical techniques to approximate solutions of system of first-order initial-value problems [13].Also, it is the basic for other methods.

To clarify the previous steps when we use Euler's method to approximate solutions of a Real Constant-Coefficient Second-Order Linear Homogeneous WFC-IVP

The first IVP(5): $\begin{cases} aY_0'' + bY_0' + cY_0 = 0 \\ Y_0(\alpha_0) = \nu_0, Y_0'(\alpha_0) = \omega_0 \end{cases} \quad ; \alpha_0 \le X_0 \le \beta_0$	The second IVP(6): $\begin{cases} aY_1'' + bY_1' + cY_1 = 0 \\ Y_1(\alpha_1) = \nu_1, Y_1'(\alpha_1) = \omega_1 \end{cases} \quad ; \alpha_1 \le X_1 \le \beta_1$
The first system (7): $\begin{cases} Y_0' = Z_0 \\ Z_0' = \frac{-b}{a} Z_0 - \frac{c}{a} Y_0 \\ Y_0(\alpha_0) = \nu_0, Z_0(\alpha_0) = \omega_0 \end{cases} \quad ; \alpha_0 \le X_0 \le \beta_0$	The second system (8): $\begin{cases} Y_1' = Z_1 \\ Z_1' = \frac{-b}{a} Z_1 - \frac{c}{a} Y_1 \\ Y_1(\alpha_1) = \nu_1, Z_1(\alpha_1) = \omega_1 \end{cases} \quad ; \alpha_1 \le X_1 \le \beta_1$
Euler's Algorithm in R	
INPUT: Endpoints α_0, β_0; integer N; initial conditions ν_0, ω_0. STEP1: Set $h_0 = \frac{\lvert\beta_0 - \alpha_0\rvert}{N}$; $X_0[0] = \alpha_0$; $S_0[0] = \nu_0$; $T_0[0] = \omega_0$. STEP2: For $i = 0,1,2, \cdots, N$ run Steps 3, 4. STEP3: $S_0[i+1] = S_0[i] + h_0\, T_0[i]$ $T_0[i+1] = T_0[i] + h_0\, f_1(X_0[i], S_0[i], T_0[i])$ $= T_0[i] + h_0 \left[\frac{-b}{a} T_0[i] - \frac{c}{a} S_0[i]\right]$ $X_0[i] = X_0[0] + ih_0$. STEP4: OUTPUT: $(X_0[i], S_0[i], T_0[i])$, Absolute Errors: $e_0[i] = \lvert Y_{0i} - S_0[i]\rvert$; $E_0[i] = \lvert Y_{0i}{}' - T_0[i]\rvert$.	INPUT: Endpoints α_1, β_1; integer N; initial condition ν_1, ω_1. STEP1: Set $h_1 = \frac{\lvert\beta_1 - \alpha_1\rvert}{N}$; $X_1[0] = \alpha_1$; $S_1[0] = \nu_1$; $T_1[0] = \omega_1$. STEP2: For $i = 0,1,2, \cdots, N$ run Steps 3, 4. STEP3: $S_1[i+1] = S_1[i] + h_1\, T_1[i]$ $T_1[i+1] = T_1[i] + h_1\, f_2(X_1[i], S_1[i], T_1[i])$ $= T_1[i] + h_1 \left[\frac{-b}{a} T_1[i] - \frac{c}{a} S_1[i]\right]$ $X_1[i] = X_1[0] + ih_1$. STEP4: OUTPUT: $(X_1[i], S_1[i], T_1[i])$, Absolute Errors: $e_1[i] = \lvert Y_{1i} - S_1[i]\rvert$; $E_1[i] = \lvert Y_{1i}{}' - T_1[i]\rvert$.

The WFC-IVP (4) $\begin{cases} aY'' + bY' + cY = 0 \\ Y(\alpha) = \nu, Y'(\alpha) = \omega \end{cases}$; $\alpha \le X \le \beta$
Euler's Algorithm in F_J
INPUT: Endpoints α, β; integer N; initial conditions ν, ω. Step1: Set h-the step size for each $i = 1,2,\cdots,N$, as $h = \varphi^{-1}(h_0, h_1) = \frac{1}{N}\varphi^{-1}(\lvert\beta_0 - \alpha_0\rvert, \lvert\beta_1 - \alpha_1\rvert) = \frac{\lvert\beta - \alpha\rvert}{N}$; $X[0] = \varphi^{-1}(X_0[0], X_1[0]) = \varphi^{-1}(\alpha_0, \alpha_1) = \alpha$, $S[0] = \varphi^{-1}(S_0[0], S_1[0]) = \varphi^{-1}(\nu_0, \nu_1) = \nu$, $T[0] = \varphi^{-1}(T_0[0], T_1[0]) = \varphi^{-1}(\omega_0, \omega_1) = \omega$. Step2: For $i = 1, 2, \ldots, N$ run Steps 3 & 4: Step3: Set $S_{i+1} = \varphi^{-1}(S_0[i+1], S_1[i+1])$ $= \varphi^{-1}(S_0[i], S_1[i]) + \varphi^{-1}[(h_0, h_1)(T_0[i], T_1[i])] = S_i + h\,T[i]$ $T_{i+1} = \varphi^{-1}(T_0[i+1], T_1[i+1])$ $= \varphi^{-1}(T_0[i], T_1[i]) + \varphi^{-1}[(h_0, h_1)\left(\frac{-b}{a}T_0[i] - \frac{c}{a}S_0[i], \frac{-b}{a}T_1[i] - \frac{c}{a}S_1[i]\right)]$ $= T_i + h\,(\frac{-b}{a}T[i] - \frac{c}{a}S[i])$ $X[i] = \varphi^{-1}(X_{0i}, X_{1i}) = \frac{1}{2}\left[X_0[0] + X_1[0]\right] + \frac{1}{2\sqrt{t}}J\left[X_1[0] - X_0[0]\right] + ih = \alpha + ih$ Step4: OUTPUT approximation S_i to $Y[i]$ and T_i to $Y'[i]$ at the $N+1$ values of $X[i]$. The exact (analytical) solution of a WFC-IVP is $Y_i = Y[i]$. Absolute Errors: $e[i] = \varphi^{-1}(e_0[i], e_1[i])$ and $E[i] = \varphi^{-1}(E_0[i], E_1[i])$.

4 A Model of the Second-Order Real Constant-Coefficient Linear Homogeneous WFC-IVPs with fixed step and real initial conditions

In this work, we study the case when $h = h_0 = h_1$(i.e., fixed step for (4),(5)&(6));

where the borders of X are $\alpha = \varphi^{-1}(\alpha_0, \alpha_1)$ and $\beta = \varphi^{-1}(\beta_0, \beta_1)$,

$\alpha < \beta$, and

$$|\beta - \alpha| = |\beta_0 - \alpha_0| = |\beta_1 - \alpha_1|.$$

Remark. In case $h = h_0 = h_1$and $\nu, \omega \in \boldsymbol{R}$, we will get the same results for approximate solutions of the WFC-IVP(4) and its related IVP(5) and IVP(6):

1) $X[i] = X_0[i] = X_1[i]$; $i = 1,2,\cdots,N$.

2) $S[i] = S_0[i] = S_1[i]$; $i = 1,2,\cdots,N$.

3) $T[i] = T_0[i] = T_1[i]$; $i = 1,2,\cdots,N$.

Proof:

Since $\alpha, \beta, \nu, \omega \in \boldsymbol{R}$,

$\alpha = \varphi^{-1}(\alpha_0, \alpha_1) = \xi + 0J \in \boldsymbol{R} \Rightarrow \alpha = \alpha_0 = \alpha_1 = \xi$,

$\beta = \varphi^{-1}(\beta_0, \beta_1) = \delta + 0J \in \boldsymbol{R} \Rightarrow \beta = \beta_0 = \beta_1 = \delta$,

$\nu = \varphi^{-1}(\nu_0, \nu_1) = \tau + 0J \in \boldsymbol{R} \Rightarrow \nu = \nu_0 = \nu_1 = \tau$,

$\omega = \varphi^{-1}(\omega_0, \omega_1) = \sigma + 0J \in \boldsymbol{R} \Rightarrow \omega = \omega_0 = \omega_1 = \sigma$.

Then,

1) $X_0[i] = \alpha_0 + ih_0 = \xi + ih = X_1[i] = X[i]$.

And since $S_0[0] = S_1[0] = S[0] = \nu = \tau$ & $T_0[0] = T_1[0] = T[0] = \omega = \sigma$, we get

2) $S_0[1] = S_0[0] + h_0\, T_0[0] = \tau + h\,\sigma = S_1[1] = S[1]$
$\Rightarrow \cdots \Rightarrow S_0[N] = S_1[N] = S[N]$.

3) $T_0[1] = T_0[0] + h_0\left[\frac{-b}{a}T_0[0] - \frac{c}{a}S_0[0]\right] = T_1[0] + h_1\left[\frac{-b}{a}T_1[0] - \frac{c}{a}S_1[0]\right] = T_1[1] = T[1]$
$\Rightarrow \cdots \Rightarrow T_0[N] = T_1[N] = T[N]$.

Example 1. $N = 10$

$$\begin{cases} Y'' + 5Y' + 6Y = 0 \\ Y(0) = 1,\ \ Y'(0) = 0 \end{cases} ;0 \le X \le 0.1\,; X, Y \in \boldsymbol{F_J} \qquad \textbf{(9)}$$

This is a real constant-coefficient second-order linear homogeneous weak fuzzy complex initial value problem is equivalent to the two following initial value problems in $\boldsymbol{R}$,

The first IVP(10): $\begin{cases} Y_0'' + 5Y_0' + 6Y_0 = 0 \\ Y_0(0) = 1, Y_0'(0) = 0 \end{cases}$ $;0 \le X_0 \le 0.1$	The second IVP(11): $\begin{cases} Y_1'' + 5Y_1' + 6Y_1 = 0 \\ Y_1(0) = 1, Y_1'(0) = 0 \end{cases}$ $;0 \le X_1 \le 0.1$
The first system (13): $\begin{cases} Y_0' = Z_0 \\ Z_0' = -5Z_0 - 6Y_0 \\ Y_0(0) = 1, Z_0(0) = 0 \end{cases}$ $;0 \le X_0 \le 0.1$	The second system (14): $\begin{cases} Y_1' = Z_1 \\ Z_1' = -5Z_1 - 6Y_1 \\ Y_1(0) = 1, Z_1(0) = 0 \end{cases}$ $;0 \le X_1 \le 0.1$
The exact solution at the step i	
$Y_{0i} = 3e^{-2X_{0i}} - 2e^{-3X_{0i}}$	$Y_{1i} = 3e^{-2X_{1i}} - 2e^{-3X_{1i}}$
The approximate solutions by Euler's Algorithm for $N = 10$	
INPUT: $\alpha_0 = 0, \beta_0 = 1; N = 10, \nu_0 = 1, \omega_0 = 0.$ STEP1: Set $h_0 = 0.01$; $X_0[0] = 0$ $S_0[0] = 1$ $T_0[0] = 0$ STEP2: For $i = 0,1,2,\cdots,N$ run Steps 3, 4 STEP3: $S_0[i+1] = S_0[i] + 0.01T_0[i]$ $T_0[i+1] = T_0[i] + 0.01\,[-5T_0[i] - 6S_0[i]]$ $X_0[i] = X_0[0] + 0.01i$ STEP4: OUTPUT: $(X_0[i], S_0[i], T_0[i])$, Absolute Errors: $e_0[i] = \lvert Y_{0i} - S_0[i]\rvert$; $E_0[i] = \lvert Y_{0i}' - T_0[i]\rvert$.	INPUT: $\alpha_1 = 0, \beta_1 = 1, N = 10, \nu_1 = 1, \omega_1 = 0.$ STEP1: Set $h_1 = 0.01$; $X_1[0] = 0$ $S_1[0] = 1$ $T_1[0] = 0$ STEP2: For $i = 0,1,2,\cdots,N$ run Steps 3, 4 STEP3: $S_1[i+1] = S_1[i] + 0.01\,T_1[i]$ $T_1[i+1] = T_1[i] + 0.01\,[-5T_1[i] - 6S_1[i]]$ $X_1[i] = X_1[0] + 0.01i$ STEP4: OUTPUT: $(X_1[i], S_1[i], T_1[i])$, Absolute Errors: $e_1[i] = \lvert Y_{1i} - S_1[i]\rvert$; $E_1[i] = \lvert Y_{1i}' - T_1[i]\rvert$.
The WFC-IVP(9) $\begin{cases} Y'' + 5Y' + 6Y = 0 \\ Y(0) = 1,\ \ Y'(0) = 0 \end{cases}$ $;0 \le X \le 0.1$	
The approximate solutions by Euler's Algorithm for $N = 10$	

INPUT:
Endpoints α, β; integer N; initial conditions ν, ω.
Step1: Set h-the step size for each $i = 1,2,\cdots, N$, as
$h = h_0 = h_1 = 0.01$
$X[0] = X_0[0] = X_1[0] = 0$
$S[0] = S_0[0] = S_1[0] = 1$
$T[0] = T_0[0] = T_1[0] = 0$
Step2: For $i = 1, 2, \dots, N$ run Steps 3 & 4:
Step3: Set $S_{i+1} = S_0[i+1] = S_1[i+1]$
$T_{i+1} = T_0[i+1] = T_1[i+1]$
$X_i = X_{0i} = X_{1i}$
Step4: OUTPUT approximation S_i to $Y[i]$ and T_i to $Y'[i]$ at the $N+1$ values of $X[i]$.
- Absolute Errors:

$e[i] = \varphi^{-1}(e_0[i], e_1[i]) = \varphi^{-1}(|Y_{0i} - S_0[i]|, |Y_{1i} - S_1[i]|)$
$E[i] = \varphi^{-1}(E_0[i], E_1[i]) = \varphi^{-1}(|Y_{0i}{}' - T_0[i]|, |Y_{1i}{}' - T_1[i]|)$

Using Python, we get the following results,

Table 5. The outputs of Euler's Algorithm for IVPs (10) & (11)

i	$X_{0i} = X_{1i}$	$S_0[i] = S_1[i]$	$Y_{0i} = Y_{1i}$	$e_0[i] = e_1[i]$	$T_0[i] = T_1[i]$	$Y'_{0i} = Y_{1i}{}'$	$E_0[i] = E_1[i]$
0	0.0	1.0	1.0	0.0	0.0	0.0	0.0
1	0.01	1.0	0.999704952	0.000295047	−0.06	−0.058518838	0.001481161
2	0.02	0.9994	0.998839250	0.000560749	−0.116999999	−0.114149433	0.002850566
3	0.03	0.99823	0.997431230	0.000798769	−0.171114	−0.167000089	0.004113910
4	0.04	0.9965188599	0.995508165	0.001010694	−0.222452099	−0.217175458	0.005276641
5	0.05	0.9942943389	0.993096301	0.001198037	−0.271120626	−0.264776649	0.006343976
6	0.06	0.99158313273	0.990220887	0.001362245	−0.317222255	−0.309901351	0.007320903
7	0.07	0.98841091017	0.986906214	0.001504695	−0.360856130	−0.352643936	0.008212194
8	0.08	0.98480234886	0.983175644	0.001626704	−0.402117978	−0.393095567	0.009022411
9	0.09	0.98078116908	0.979051645	0.001729523	−0.441100220	−0.431344302	0.009755918
10	0.1	0.97637016687	0.974555817	0.001814349	−0.477892079	−0.467475194	0.010416885

Figure 1. The results of approximate solutions, and exact solutions for IVPs (10) & (11).

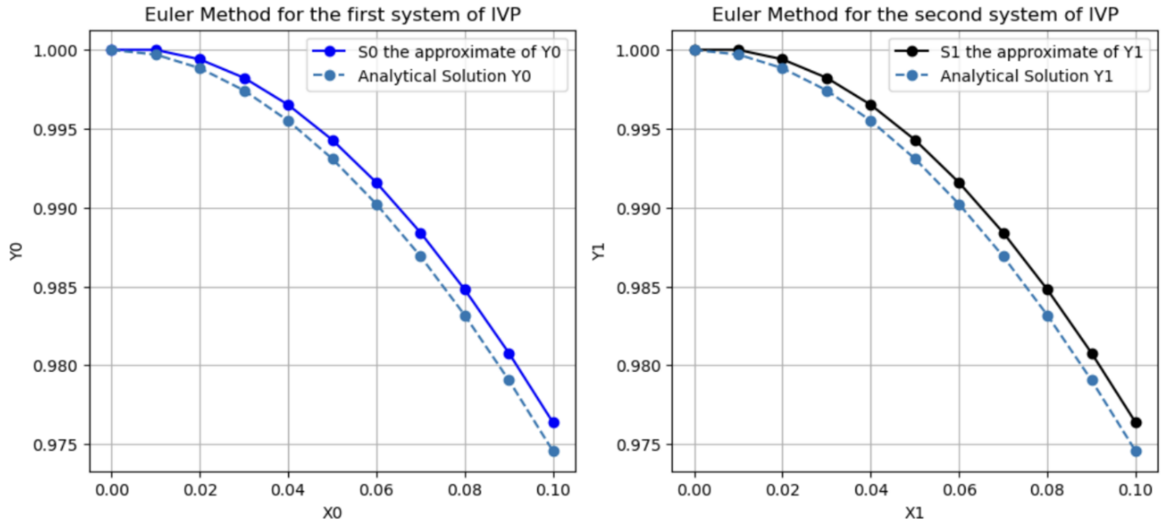
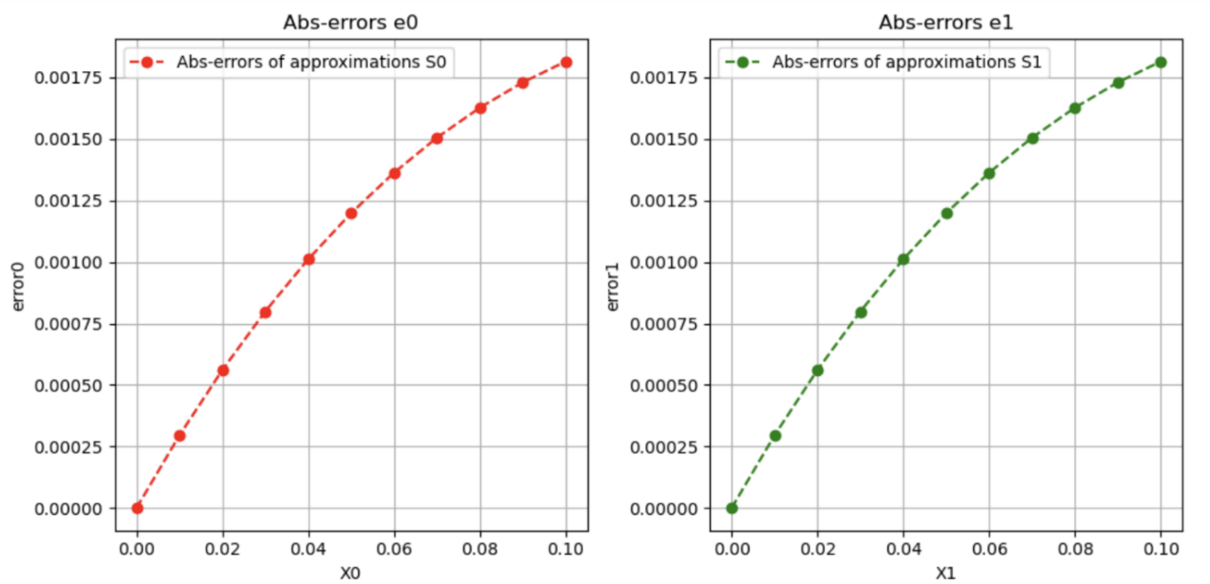


Figure 2. The results of absolute errors of the approximations $S_0[i]$ & $S_1[i]$.



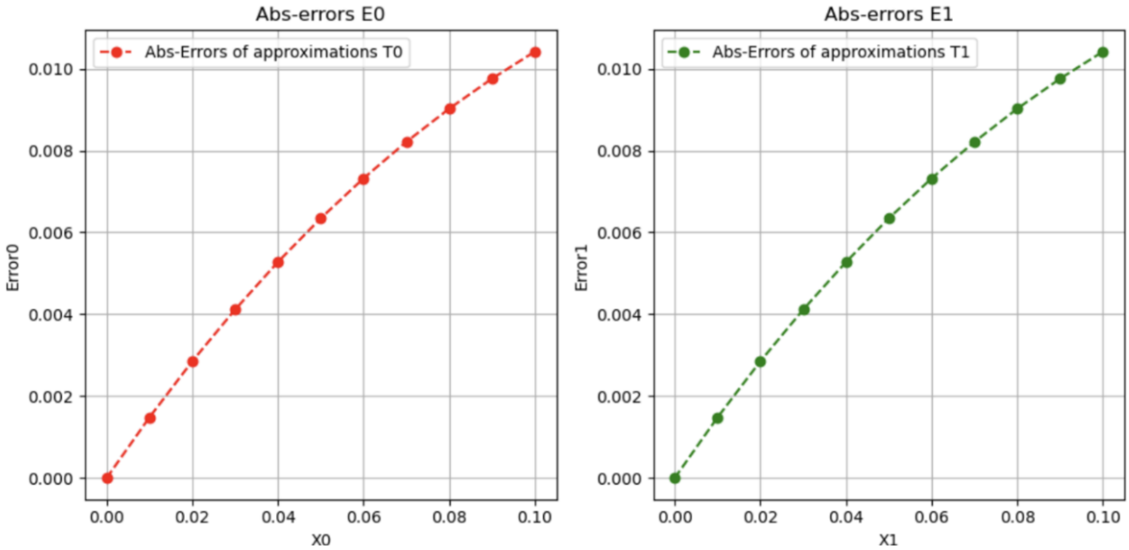


Figure 3. The results of absolute errors of the approximations $T_0[i]$& $T_1[i]$.

Table 6. The outputs of Euler's Algorithm of WFC-IVP (9)

i	X_i	$S[i] = S_0[i] = S_1[i]$	$e[i] = \varphi^{-1}(e_0[i], e_1[i])$	$T[i] = T_0[i] = T_1[i]$	$E[i] = \varphi^{-1}(E_0[i], E_1[i])$
0	0.0	1.0	0.0	0.0	0.0
1	0.01	1.0	0.000295047	−0.06	0.001481161
2	0.02	0.9994	0.000560749	−0.116999999	0.002850566
3	0.03	0.99823	0.000798769	−0.171114	0.004113910
4	0.04	0.9965188599	0.001010694	−0.222452099	0.005276641
5	0.05	0.9942943389	0.001198037	−0.271120626	0.006343976
6	0.06	0.99158313273	0.001362245	−0.317222255	0.007320903
7	0.07	0.98841091017	0.001504695	−0.360856130	0.008212194
8	0.08	0.98480234886	0.001626704	−0.402117978	0.009022411
9	0.09	0.98078116908	0.001729523	−0.441100220	0.009755918
10	0.1	0.97637016687	0.001814349	−0.477892079	0.010416885

Note that we got the same results for approximate solutions of the WFC-IVP (4) and its related IVPs (5) and (6).

5 Conclusion

In this paper, we have solved a model of the second-order Weak Fuzzy Complex- initial value problem (WFC-IVP) numerically. Using a special isomorphism transformation function, the WFC-IVP could be written as two second-order IVPs in ***R*****,** transforming into systems of first-order IVPs. Hence, we have focused on solving second-order real constant-coefficient linear homogeneous WFC-initial value problems with fixed step and real initial conditions using Euler's method. The steps of finding the approximations have been explained in examples with tables and diagrams of results.

In the future, we aim to solve WFC-IVPs using another methods and study different models of weak fuzzy complex-initial value problems.

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Appendix

This Appendix includes the instructions by Python for the results of solving IVPs (5).

jupyter article٢WFC-IVP ٢nd-order Last Checkpoint: ٢٠٢٥/٥/١ (autosaved)

File Edit View Insert Cell Kernel Widgets Help

Run Code

In [1]:

```
##### For the first IVP in R: ---------------------------(for outputs:(X0, S, T))
import numpy as np
import matplotlib.pyplot as plt
#The corresponding function:
def F1(u, s, t):
    return -5*t-6*s
#The Parameters
u0 = 0       # Start time
s0 = 1       # Initial condition Y0[0]=s0
t0 = 0       # Initial condition Y'0[0]=t0
uN = 0.1       # End time
N = 10       # The number of iterations
print ('Euler Method Results')
#STEP1
h = float ((uN - u0) / N) #the step size
print ('h=', h)
print ('The initial values i=0  : (X0[0]=',u0,',S[0]=',s0,',T[0]=',t0,')')
def euler_method1(F1 , u0, s0, t0, N, uN):
    u = np.linspace(u0, uN, N + 1)
    s = np.zeros(N + 1)
    t = np.zeros(N + 1)
    s[0] = s0
    t[0] = t0
#STEP2
    for i in range (N):
        #STEP3&4
        s [i + 1] = s [i] + h * t[i]
        t [i + 1] = t [i] + h * F1(u[i], s[i],t[i])
        print (' i=',i,', (X0[',i+1,']=',u[i+1],', S[',i+1,']=', s[i+1],' ,T[',i+1,']=', t[i+1],')')
    return u, s, t
# Solve using Euler's method
u, s, t = euler_method1(F1 , u0, s0, t0, N, uN)
#EXACT solution
# Compute the analytical solution
def analytical_solution1(u):
    return np.exp(u)
print ('The exact solutions')
for i in range(N+1):
    Y0 = 3*analytical_solution1(-2*u)-2*analytical_solution1(-3*u)
    print ('Y0[',i,']=',Y0[i])
    T0 = -6*analytical_solution1(-2*u)+6*analytical_solution1(-3*u)
    print ('Z0[',i,']=',T0[i])

# Calculate the error
error1 = np.abs(Y0 - s)
Error1 = np.abs(T0 - t)
print ('Abs-Errors of Y')
for i in range (N+1):
        print ('e[',i,']=', error1[i])
print ('Abs-Errors of Z')
for i in range (N+1):
        print ('E[',i,']=', Error1[i])
```